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EQUATIONS OF PERTURBED MOTION OF A BODY WITH A THIN-WALLED ELASTIC SHELL PARTIALLY FILLED WITH A LIQUID

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Linear equations of perturbed motion of a thin-walled elastic shell partially filled with a heavy compressible fluid considered in the acoustic approximation are derived; the principal [force] vector and the principal moment of the reactions exerted by the shell on the "carrying body" are determined. Perturbed motion with small vibrations is characterized by the displacement of a certain point attached to the rigid shell fastening contour, by rotation relative to this point, and by elastic displacements expressed as an expansion in the proper vibration modes of the fastened fluid-containing shell. The natural frequencies and vibration modes of a fluid-containing shell are determined by means of a variational principle.

Allowance for the compressibility of the fluid makes it possible to consider vibrations in the acoustic frequency spectrum. Moreover, calculations show that it may be necessary to make allowance for it in calculating the lower frequencies of the elastic vibrations of the shell, e. g. of the axisymmetric vibrations of relatively thick shells of revolution. Allowance for gravity is necessary in considering vibrations in the frequency spectrum of gravitational surface waves and vibrations of flexible fluid-containing shells.

1. Formulation of the problem. Let us consider the perturbed motion of a "carrying" body with an attached thin-walled elastic shell containing an ideal compressible fluid. The shell is fastened to the body along the contour Γ which we assume to be nondeformable.

In order to avoid limiting ourselves to some specific model of a carrying body (e.g. an absolutely rigid solid, we propose to isolate the fluid-containing shell along the contour Γ, write the equations of perturbed motion for the shell and fluid, and determine the principal vector $\mathbf{T}$ and principal moment $\mathbf{H}$ exerted by the shell on the carrying body along the contour Γ during perturbed motion. This will enable us to write out the equations of perturbed motion of the carrying body with allowance for the reactions $\mathbf{T}$ and $\mathbf{H}$, and thus to obtain the closed system of equations of perturbed motion.

The perturbed motion of a fluid-containing shell can be characterized by the vector $\mathbf{u}$ of small displacements of the shell and by the potential Φ of small displacements of the fluid in the coordinate system $Ox_1x_2x_3$; the latter experiences translational motions which coincide with the unperturbed motions of the body. We make the axis Ox_1 perpendicular to the unperturbed surface σ of the fluid; this makes the direction of Ox_1 opposite to that of the body-force vector $\mathbf{g}$.

The equations of perturbed motion of a fluid-containing shell are obtainable with the aid of the Lagrange variational principle

$$\delta\Pi + \iint_S m\,\mathbf{u}^{\cdot\cdot}\delta\mathbf{u}\,dS + \rho\iiint_\tau \nabla\Phi^{\cdot\cdot}\delta\nabla\Phi\,d\tau - \delta A = 0 \tag{1.1}$$

$$\Pi = \Pi_0 + \frac{\rho g}{2}\iint_\sigma \left(\frac{\partial\Phi}{\partial\nu}\right)^2 d\sigma + \frac{\rho c^2}{2}\iiint_\tau (\Delta\Phi)^2\,d\tau \tag{1.2}$$

Here Π is the potential energy of the shell and the compressible fluid system; m, ρ, c are the specific mass of the shell (its mass divided by the area of its middle surface), the density of the fluid, and the velocity of sound in the latter, respectively; S and σ are the shell surface and the free surface of the fluid; τ is the volume occupied by the fluid; $\boldsymbol{\nu}$ is the unit vector of the exterior normal of the surface bounding the volume τ; $gH/c^2 \ll 1$, $\rho \approx \text{const}$.

The quantity Π_0 in (1.2) is the potential energy of shell deformation in perturbed motion with allowance for the forces which arise in its middle surface during unperturbed motion; in addition, Π_0 includes that part of the potential energy of the body forces of the fluid which depends solely on the shell displacements and can be determined by assuming that the fluid surface is fixed. The potential energy of the body forces of the fluid associated with displacements of the free surface is represented by the second term in expression (1.2); the third term allows for the potential energy of compression of the fluid in the acoustic approximation.

The variation of the work performed by the specific surface load $\mathbf{q}$ applied to the shell with allowance for the reactions $\mathbf{T}$ and $\mathbf{H}$ between the shell and carrying body is given by

$$\delta A = \iint_S \mathbf{q}\delta\mathbf{u}\,ds - \mathbf{T}\delta\mathbf{u}_0 - \mathbf{H}\delta\boldsymbol{\theta}_0 \tag{1.3}$$

where $\mathbf{u}_0$ and $\boldsymbol{\theta}_0$ are the displacement vector of some point O' rigidly attached to the contour Γ and the vector of a small rotation about this point characterizing the motion of the undeformed shell.

Equation (1.1) can be used only in the event of fulfillment of kinematic boundary conditions at the shell edges and of the kinematic boundary condition

$$\partial\Phi/\partial\mathbf{v} - \mathbf{vu} = 0 \text{ at } S_0 \qquad (1.4)$$

at the wet surface S_0 of the shell; the continuity condition

$$\Delta\Phi = 0 \quad \text{в } \tau \qquad (1.5)$$

must be fulfilled in the case of an incompressible fluid $(c \to \infty)$.

2. Conditions of orthogonality of the natural vibration modes of a fluid-containing shell. Let us suppose that we know the natural frequencies ω_n and proper vibration modes $\mathbf{u}_n$, Φ_n of a fluid-containing shell, and that these frequencies and modes satisfy the equations

$$\Delta\Phi_n + \frac{\omega^2{}_n}{c^2}\Phi_n = 0 \text{ in } \tau$$

$$\frac{\partial\Phi_n}{\partial\mathbf{v}} - \mathbf{vu}_n = 0 \quad \text{on } S_0, \qquad g\frac{\partial\Phi_n}{\partial\mathbf{v}} - \omega^2{}_n\Phi_n = 0 \quad \text{on } \sigma$$

$$L(\mathbf{u}_n) - \omega^2{}_n m\mathbf{u}_n - \omega^2{}_n \varepsilon\rho\Phi_n\mathbf{v} = 0 \quad \text{on } S \qquad (2.1)$$

and the corresponding boundary conditions at the shell edges. The symbol $L(\cdots)$ denotes the linear selfadjoint differential operator of the shell equations associated with the potential energy Π_0

$$\delta\Pi_0 = \iint_S L(\mathbf{u})\,\delta\mathbf{u}\,dS \qquad (2.2)$$

provided $\mathbf{u}$ satisfies all the boundary conditions at the shell edges; $\varepsilon = 1$ at S_0 and $\varepsilon = 0$ at $S - S_0$.

We derive the orthogonality conditions by means of the Lagrange principle for $\delta A = 0$,

$$\delta\Pi + \iint_S m\mathbf{u}^{\cdot\cdot}\delta\mathbf{u}\,dS + \rho\iiint_\tau \nabla\Phi^{\cdot\cdot}\delta\nabla\Phi d\tau = 0 \qquad (2.3)$$

Let us suppose that the free vibrations take the form of a superposition of the nth and mth proper modes,

$$\mathbf{u} = q_n\mathbf{u}_n + q_m\mathbf{u}_m, \qquad \Phi = q_n\Phi_n + q_m\Phi_m \qquad (2.4)$$

$$q_n(t) = q_n^\circ \cos(\omega_n t + \gamma_n), \qquad q_m(t) = q_m^\circ \cos(\omega_m t + \gamma_m)$$

Substituting (2.4) into (2.3) and taking account of the arbitrariness of the variations δq_n and δq_m, we obtain the two equations

$$\sum_{=n,m}\left[k_{ij} - \omega_j^2\left(\iint_S m\mathbf{u}_i\mathbf{u}_j dS + \rho\iiint_\tau \nabla\Phi_i\nabla\Phi_j d\tau\right)\right]q_j = 0 \quad (i = n, m) \qquad (2.5)$$

where k_{ij} are the coefficients of the expansion of the potential energy

$$\Pi = {}^1/_2\sum_i\sum_j k_{ij}q_iq_j$$

Let us set first $q_n \neq 0$, $q_m \equiv 0$ and then $q_n \equiv 0$, $q_m \neq 0$, in Eqs. (2.5). This yields

$$k_{ij} - \omega_j^2\left(\iint_S m\mathbf{u}_i\mathbf{u}_j dS + \rho\iiint_\tau \nabla\Phi_i\nabla\Phi_j d\tau\right) = 0 \qquad (2.6)$$

$$(i = n, m; \; j = n, m)$$

Writing out Eq. (2.6) for $i = n,\ j = m$, subtracting Eq. (2.6) for $i = m,\ j = n$ from the result, and taking advantage of the symmetry of the coefficients k_{nm} ($k_{nm} = k_{mn}$), we obtain the conditions of orthogonality of the proper modes for $n \neq m$,

$$\iint_S m\mathbf{u}_n\mathbf{u}_m dS + \rho \iiint_\tau \nabla\Phi_n \nabla\Phi_m d\tau = 0 \qquad (n \neq m) \tag{2.7}$$

In addition, Eq. (2.6) for $i = n,\ j = m\ (n \neq m)$ implies that $k_{nm} = 0$, or, with allowance for (1.2), (2.2), that

$$\iint_S L(\mathbf{u}_n)\mathbf{u}_m dS + \rho g \iint_\sigma \frac{\partial\Phi_n}{\partial\nu} \cdot \frac{\partial\Phi_m}{\partial\nu} d\sigma + \rho c^2 \iiint_\tau \Delta\Phi_n \Delta\Phi_m d\tau = 0 \tag{2.8}$$
$$(n \neq m)$$

For $i = j = n$ Eq. (2.6) gives us

$$k_{nn} = \omega_n{}^2 m_n, \qquad m_n = \iint_S m\mathbf{u}_n{}^2 dS + \rho \iiint_\tau (\nabla\Phi_n)^2 d\tau \tag{2.9}$$

for the virtual mass associated with the nth natural vibration mode.

Expressions (2.7), (2.8), (2.9) can be replaced by several equivalent relations obtainable by way of Eqs. (2.1) and Green's transformation formula for a volume integral.

3. Equations of motion of a fluid-containing shell. The displacement vector of points of the middle surface of the shell and the fluid particle displacement potential can be written as

$$\mathbf{u} = \mathbf{u}_0 + \theta_0 \times \mathbf{r}' + \sum_{n=1}^{\infty} q_n\mathbf{u}_n, \qquad \Phi = \mathbf{u}_0\mathbf{r}' + \theta_0\Psi + \sum_{n=1}^{\infty} q_n\Phi_n \tag{3.1}$$
$$(\mathbf{r}' = \mathbf{r} - \mathbf{r}_0 = x_1'\mathbf{i}_1 + x_2'\mathbf{i}_2 + x_3'\mathbf{i}_3)$$

Here $\mathbf{r}$ and $\mathbf{r}_0$ are the radius vectors of the point in question and of the point O'; $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$ are the unit vectors of the coordinate system $Ox_1x_2x_3$; x_1', x_2', x_3' are the coordinates measured from the point O'; $\mathbf{u}_n$, Φ_n are the proper vibration modes of the fluid-containing shell fastened along the contour, $\mathbf{u}_{n|\Gamma} = 0$; $q_n(t)$ are the generalized coordinates characterizing the deformations of the shell and fluid and the wave motions of the free surface of the fluid.

We assume that the vector function $\Psi = \Psi_1\mathbf{i}_1 + \Psi_2\mathbf{i}_2 + \Psi_3\mathbf{i}_3$ describing the motion of the fluid during rotation of the undeformed shell is harmonic in the domain τ and generally arbitrary at the free surface σ. The wave equation in τ and the dynamic boundary condition at σ are satisfied by virtue of the generalized coordinates q_n, since the latter are the coefficients of the system of functions Φ_n complete in τ and at $S_0 + \sigma$. This means that Ψ satisfies the following equation and boundary condition:

$$\Delta\Psi = 0 \text{ in } \tau, \qquad \frac{\partial\Psi}{\partial\nu} = \mathbf{r}' \times \nu \quad \text{on } S_0 \tag{3.2}$$

The function Ψ at the surface σ can be subjected to one of the following conditions: 1) as in [1, 2] the function Ψ consists of Zhukovskii potentials describing a motion of the fluid in which the free surface remains flat and rotates together with the undeformed shell; here $\partial\Psi / \partial\nu = \mathbf{r}' \times \nu$ at σ; (2) as in [3], the function Ψ describes a motion during rotation of the undeformed shell such that the free surface of the fluid remains plane and parallel to the unperturbed free surface, $\partial\Psi / \partial\nu = \mathbf{c}$ at σ, where

$\mathbf{c} = c_1\mathbf{i}_1 + c_2\mathbf{i}_2 + c_3\mathbf{i}_3$ can be determined from the condition of a constant fluid volume,

$$c_1 = 0, \qquad c_2 = \frac{1}{S_\sigma}\iint_\sigma x_3' d\sigma, \qquad c_3 = -\frac{1}{S_\sigma}\iint_\sigma x_2' d\sigma$$

3) in calculating "rapid" motions of the shell when practically no lower modes of the gravitational waves at the free surface are excited and when the influence of g is negligible, it is convenient to subject the function Ψ to the condition $\Psi = 0$ on σ.

We begin by calculating the variation of the work performed by the inertial forces of the shell and fluid. Making use of expansions (3.1), recalling (3.2) and orthogonality conditions (2.7), and carrying out the appropriate transformations, we obtain

$$\iint_S m\mathbf{u}^{\cdot\cdot}\delta\mathbf{u}dS + \rho\iiint_\tau \nabla\Phi^{\cdot\cdot}\delta\nabla\Phi d\tau = \left[\mathbf{u}_0^{\cdot\cdot}m_0 + \theta^{\cdot\cdot}{}_0\mathbf{S} + \sum_{n=1}^{\infty} q_n^{\cdot\cdot}\mathbf{m}_{0n}\right]\delta\mathbf{u}_0 +$$
$$+\left[\mathbf{u}_0^{\cdot\cdot}\mathbf{S}' + \theta_0^{\cdot\cdot}\mathbf{J} + \sum_{n=1}^{\infty} q^{\cdot\cdot}{}_n\lambda_{0n}\right]\delta\theta_0 + \sum_{n=1}^{\infty}\left[\mathbf{u}_0^{\cdot\cdot}\mathbf{m}_{0n} + \theta_0^{\cdot\cdot}\lambda_{0n} + q_n^{\cdot\cdot}m_n\right]\delta q_n \qquad (3.3)$$

Here m_0 is the mass of the shell and fluid; $\mathbf{S}$ and $\mathbf{J}$ are the tensor of static moments and the inertia tensor, respectively ($\mathbf{S}'$ is the associated tensor),

$$\mathbf{S} = \begin{bmatrix} 0 & S_3 & -S_2 \\ -S_3 & 0 & S_1 \\ S_2 & -S_1 & 0 \end{bmatrix} + [S_{jk}]$$

$$S_k = \iint_S m x_k dS, \qquad S_{jk} = \rho\iint_{S_0+\sigma} x_j' \frac{\partial\Psi_k}{\partial\nu} dS$$

$$\mathbf{J} = [J_{jk}^\circ] + [J_{jk}]$$

$$J_{jk}^\circ = \iint_S m\left(\delta_{jk}\mathbf{r}'^2 - x_j'x_k'\right) dS, \qquad J_{jk} = \rho\iint_{S_0+\sigma} \Psi_j \frac{\partial\Psi_k}{\partial\nu} dS$$

(δ_{jk} is the Kronecker delta; $j, k = 1, 2, 3$)

$$\mathbf{m}_{0n} = \iint_S m\mathbf{u}_n dS + \rho\iint_{S_0+\sigma} \Phi_n \boldsymbol{\nu} dS$$

$$\lambda_{0n} = \iint_S m(\mathbf{r}'\times\mathbf{u}_n) dS + \rho\iint_{S_0+\sigma} \Phi_n \frac{\partial\Psi}{\partial\nu} dS \qquad (3.4)$$

Now let us write out the expression for the variation of the potential energy of the shell and fluid. Substituting expansions (3.1) into (1.2) and recalling orthogonality condition (2.8) for the proper modes, we obtain

$$\delta\Pi = \left[\theta_0\mathbf{C} + \sum_{n=1}^{\infty} q_n\varkappa_{0n}\right]\delta\theta_0 + \sum_{n=1}^{\infty}\left[\theta_0\varkappa_{0n} + q_n\omega_n^2 m_n\right]\delta q_n \qquad (3.5)$$

where

$$\mathbf{C} = [C_{jk}], \qquad C_{jk} = \rho g\iint_\sigma \frac{\partial\Psi_j}{\partial\nu}\frac{\partial\Psi_k}{\partial\nu} d\sigma + \frac{\partial^2\Pi_0}{\partial\theta_{0j}\partial\theta_{0k}} \qquad (3.6)$$

$$\varkappa_{0n} = \rho g \iint\limits_{\sigma} \frac{\partial \Phi_n}{\partial \nu} \frac{\partial \Psi}{\partial \nu} d\sigma + \frac{\partial^2 \Pi_0}{\partial \theta_0 \partial q_n} \qquad \text{(cont.)}$$

With allowance for (3.1) and for the fact that $\mathbf{u}_n \mid \Gamma = 0$, we can rewrite expression (1.3) for δA as

$$\delta A = (\mathbf{P} - \mathbf{T}) \delta \mathbf{u}_0 + (\mathbf{M} - \mathbf{H}) \delta \theta_0 + \sum_{n=1}^{\infty} Q_n \delta q_n \tag{3.7}$$

$$\mathbf{P} = \iint\limits_{S} \mathbf{q} dS, \qquad \mathbf{M} = \iint\limits_{S} (\mathbf{r}' \times \mathbf{q}) dS, \qquad Q_n = \iint\limits_{S} \mathbf{q} \mathbf{u}_n dS \tag{3.8}$$

Substituting expressions (3.3), (3.5), (3.7) into Eq. (1.1) and equating the coefficients of the arbitrary variations $\delta \mathbf{u}_0$, $\delta \theta_0$, δq_n to zero, we obtain an expression for the principal vector and for the principal moment of all the forces exerted by the vibrating fluid-containing shell on the carrying body

$$\mathbf{T} = -\left(m_0 \mathbf{u}_0^{\cdot\cdot} + \mathbf{S} \theta_0^{\cdot\cdot} + \sum_{n=1}^{\infty} \mathbf{m}_{0n} q_n^{\cdot\cdot} \right) + \mathbf{P}$$

$$\mathbf{H} = -\left(\mathbf{S}' \mathbf{u}_0^{\cdot\cdot} + \mathbf{J} \theta_0^{\cdot\cdot} + \sum_{n=1}^{\infty} \lambda_{0n} q_n^{\cdot\cdot} + \mathbf{C} \theta_0 + \sum_{n=1}^{\infty} \varkappa_{0n} q_n \right) + \mathbf{M} \tag{3.9}$$

and also the equations of the vibrations of the fluid-containing shell in normal coordinates,

$$\mathbf{m}_{0n} \mathbf{u}_0^{\cdot\cdot} + \lambda_{0n} \theta_0^{\cdot\cdot} + m_n q_n^{\cdot\cdot} + \varkappa_{0n} \theta_0 + \omega_n^2 m_n q_n = Q_n \tag{3.10}$$

$$(n = 1, \ldots, \infty)$$

If the carrying body is an absolutely rigid solid, then it can be assumed attached to the shell, and the integration involved in computing the mass characteristics m_0, $\mathbf{S}$, $\mathbf{J}$, $\mathbf{C}$, the principal vector $\mathbf{P}$, and the principal moment of external moment $\mathbf{M}$ must be extended to the volume of the solid; in this case Eqs. (3.9) for $\mathbf{T} = \mathbf{H} = 0$ become the equations of motion of a solid carrying a thin-walled fluid-containing shell. The coefficients in Eq. (3.10) remain unchanged.

Equations (3.9), (3.10) for $\mathbf{T} = \mathbf{H} = 0$ are of the same form as the equations of motion of an absolutely rigid solid with a cavity partly filled with ideal incompressible fluid [1–4], and become the latter if the shell is assumed to be nondeformable and the fluid incompressible. The coordinates q_n then describe the wave motions at the free surface.

4. A shell of revolution. Let us express the vector of displacements of the middle surface of a shell of revolution in terms of its components along the tangents to the coordinate lines φ and θ (Fig. 1) and along the exterior normal $\boldsymbol{\nu}$ to the surface at point under consideration,

$$\mathbf{u} = u\mathbf{e}_1 + v\mathbf{e}_2 + w\boldsymbol{\nu}$$

where $\mathbf{e}_1$ and $\mathbf{e}_2$ are the unit vectors of the coordinate lines φ and θ.

The potential energy Π_0 in the general case can be expressed as a sum of three components,

$$\Pi_0 = \Pi_0^{(1)} + \Pi_0^{(2)} + \Pi_0^{(3)} \tag{4.1}$$

Here $\Pi_0^{(1)}$ is the potential energy of shell deformation on the basis of the Kirchhoff hypothesis [5]; $\Pi_0^{(2)}$ is the potential energy of the forces in the middle surface which arise in unperturbed motion; $\Pi_0^{(3)}$ is the potential energy of the body forces of the fluid during perturbed motion in the case of a stationary free surface. Following [6], we express

$\Pi_0^{(2)}$ as

$$\Pi_0^{(2)} = \frac{1}{2} \iint\limits_S [N_1^\circ (\vartheta_1^2 + \vartheta_{12}^2) + N_2^\circ (\vartheta_2^2 + \vartheta_{12}^2) + 2N_{12}^\circ \vartheta_1 \vartheta_2]\, dS \tag{4.2}$$

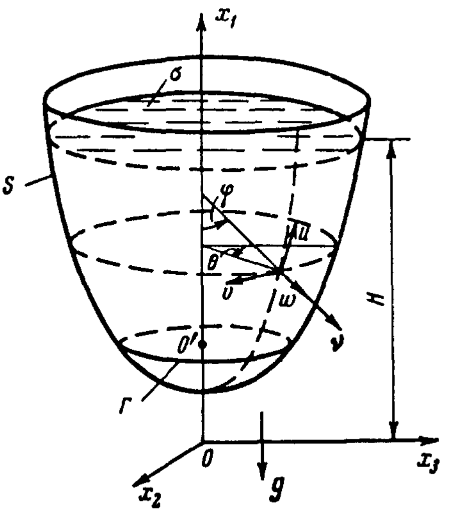

Fig. 1

For a shell of revolution we have

$$\vartheta_1 = \frac{u}{R_1} - \frac{1}{R_1} \cdot \frac{\partial w}{\partial \varphi},$$

$$\vartheta_2 = \frac{v}{R_2} - \frac{1}{R} \frac{\partial w}{\partial \theta}, \qquad R = R_2 \sin\varphi, \quad dS = R_1 R\, d\varphi\, d\theta$$

$$\vartheta_{12} = \frac{1}{2}\left[\frac{1}{R_1}\left(\frac{\partial v}{\partial \varphi} - \frac{\partial R_1}{R \partial \theta} u\right) - \frac{1}{R}\left(\frac{\partial u}{\partial \theta} - \frac{\partial R}{R_1 \partial \varphi} v\right)\right] \tag{4.3}$$

Here R_1 and R_2 are the principal radii of curvature of the middle surface of the shell.

The hydrostatic pressure acting on a shell whose axis is parallel to the axis Ox_1 is given by

$$p = \rho g\, [(H - x_1) - (u \sin\varphi - w\cos\varphi)] \tag{4.4}$$

Here x_1 is the coordinate of a point on S_0 in unperturbed motion; $x_1 = H$ at σ. The variation of the work performed by the hydrostatic pressure in unperturbed motion with allowance for the change $dS^* = dS\,(1 + \varepsilon_1 + \varepsilon_2)$ in the area of the shell elements as a result of deformation can be written as

$$\delta A_p = \iint\limits_{S_0} p\, [\delta w + \vartheta_1 \delta u + \vartheta_2 \delta v]\, (1 + \varepsilon_1 + \varepsilon_2)\, dS$$

Substituting the pressure p (4.4), the angles of rotation ϑ_1 and ϑ_2 (4.3), and the strains

$$\varepsilon_1 = \frac{1}{R_1} \frac{\partial u}{\partial \varphi} + \frac{w}{R_1}, \qquad \varepsilon_2 = \frac{1}{R} \frac{\partial v}{\partial \theta} + \frac{u}{R_2} \operatorname{ctg}\varphi + \frac{w}{R_2}$$

into this expression and retaining only the linear terms in front of the displacement variations, we obtain

$$\delta A_p = \rho g \iint\limits_{S_0} (H - x_1)\, \delta w\, dS + \iint\limits_{S_0} (p_1 \delta u + p_2 \delta v + p_\nu \delta w)\, dS \tag{4.5}$$

$$p_1 = \rho g\, (H - x_1) \frac{1}{R_1}\left(u - \frac{\partial w}{\partial \varphi}\right), \qquad p_2 = \rho g\, (H - x_1) \frac{1}{R_2}\left(v - \frac{1}{\sin\varphi} \frac{\partial w}{\partial \theta}\right)$$

$$p_\nu = \rho g \left\{-(H - x_1) \frac{1}{R_1} \frac{\partial u}{\partial \varphi} + [\sin\varphi - (H - x_1) \frac{1}{R_2} \operatorname{ctg}\varphi]\, u - (H - x_1) \frac{1}{R} \frac{\partial v}{d\theta} - \left[\cos\varphi + (H - x_1)\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\right] w\right\} \tag{4.6}$$

Here p_1, p_2, p_ν are the components of the reduced load acting on the shell due to hydrostatic pressure during perturbed motion.

Expression (4.5) becomes

$$\delta A_p = \rho g \iint_{S_0} (H - x_1)\, \delta w dS + \rho g \left[\int_0^{2\pi} (H - x_1)\, R w \delta u d\theta \right]_{x_1 = x_{10}}^{x_1 = H} - \delta \Pi_0^{(3)} \tag{4.7}$$

$$\Pi_0^{(3)} = -\frac{\rho g}{2} \iint_{S_0} \left\{ (H - x_1) \left[\frac{1}{R_1} u^2 + \frac{1}{R_2} v^2 + \left(\frac{1}{R_1} + \frac{1}{R_2} \right) w^2 + 2 \frac{\partial v}{R \partial \theta} w \right] + \right.$$
$$\left. + w^2 \cdot \cos\varphi + \frac{2}{R R_1} \frac{\partial}{\partial \varphi} [(H - x_1) R u]\, w \right\} dS \tag{4.8}$$

The second term in expression (4.7) for shells closed below at $x_1 = x_{10}$ is always equal to zero, and in this case we have

$$\delta \Pi_0^{(3)} = - \iint_{S_0} (p_1 \delta u + p_2 \delta v + p_\nu \delta w)\, dS \tag{4.9}$$

In accordance with (4.1) the operator $L\,(\ldots)$ can also be expressed in terms of three components,

$$L(\mathbf{u}) = L^{(1)}(\mathbf{u}) + L^{(2)}(\mathbf{u}) + L^{(3)}(\mathbf{u})$$

and by virtue of (4.8), (4.9)

$$L^{(3)}(\mathbf{u}) = -\varepsilon\, (p_1 \mathbf{e}_1 + p_2 \mathbf{e}_2 + p_\nu \boldsymbol{\nu})$$

is selfadjoint, as are $L^{(1)}(\mathbf{u})$ and $L^{(2)}(\mathbf{u})$.

When applied to a shell of revolution fastened axisymmetrically along the closed contour Γ with the point O' placed on the shell axis, system of equations of perturbed motion (3.9), (3.10) breaks down into equations describing the longitudinal axisymmetric vibrations, into two similar independent systems of equations describing the transverse antisymmetric vabrations in the planes Ox_1x_2 and Ox_1x_3, and into equations of asymmetric vibrations unrelated to the motion of the shell as a rigid body.

The equations of the longitudinal vibrations are

$$T_1 = -\left(m_0 \ddot{u}_{01} + \sum_{n=1}^{\infty} m_{0n1} \ddot{q}_n \right) + P_1$$

$$m_{0n1} \ddot{u}_{01} + m_n \ddot{q}_n + \omega_n^{2} m_n q_n = Q_n \qquad (n = 1, \ldots, \infty) \tag{4.10}$$

The equations of the transverse vibrations in the plane Ox_1x_3 are

$$T_3 = -\left(m_0 \ddot{u}_{03} + S_{22}^* \ddot{\theta}_{02} + \sum_{n=1}^{\infty} m_{0n3} \ddot{q}_n \right) + P_3 \tag{4.11}$$

$$H_2 = -\left(S_{22}^* \ddot{u}_{03} + I_{22}^* \ddot{\theta}_{02} + \sum_{n=1}^{\infty} \lambda_{0n3} \ddot{q}_n - g S_{22}^* \theta_{02} + \sum_{n=1}^{\infty} \varkappa_{0n3} q_n \right) + M_2$$

$$m_{0n3} \ddot{u}_{03} \quad \lambda_{0n3} \ddot{\theta}_{02} + m_n \ddot{q}_n + \varkappa_{0n3} \theta_{02} + \omega_n^{2} m_n q_n = Q_n$$
$$(n = 1, \ldots, \infty)$$

Here

$$S_{22}^* = - m_0 x_{1t}' + \rho \iint_{\sigma} \left(\frac{\partial \Psi_2}{\partial \nu} - x_3' \right) x_3' d\sigma$$

$$I_{22}^* = \iint_{S} m\, (x_1'^2 + x_3'^2)\, d\sigma + \rho \int\int_{S_0 + \sigma} \frac{\partial \Psi_2}{\partial \nu}\, \Psi_2 dS \tag{4.12}$$

(cont.)

$$\varkappa_{0n3} = \iint\limits_S \Big[N_1^\circ \frac{1}{R_1}(U_n - W_n')\cos^2\theta - N_2^\circ \frac{1}{R}(V_n \sin\varphi + W_n)\cos\varphi \sin^2\theta +$$

$$+ \frac{1}{2}(N_1^\circ + N_2^\circ)\left(\frac{1}{R}U_n + \frac{1}{R_1}V_n' + \frac{1}{R}V_n\cos\varphi\right)\sin\varphi\sin^2\theta\Big]dS +$$

$$+ \rho g \iint\limits_{S_0} [(H - x_1)(-U_n\cos^2\theta + V_n\cos\varphi\sin^2\theta) - RW_n\cos^2\theta]\,dS +$$

$$+ \rho g \iint\limits_\sigma \frac{\partial \Psi_2}{\partial \nu}\frac{\partial \Phi_n}{\partial \nu}\,d\sigma$$

Here x_{1l}' is the coordinate of the center of gravity of the fluid-containing shell measured from the point O'; $U_n(\varphi)\cos\theta$, $V_n(\varphi)\sin\theta$, $W_n(\varphi)\cos\theta$ are the displacement components of the n th proper antisymmetric vibration mode of the shell.

The equations of the asymmetric vibrations when there are two or more meridional node lines on the surfaces S and σ are

$$m_n\ddot{q}_n + \omega_n^2 m_n q_n = Q_n \qquad (n = 1, \ldots, \infty) \tag{4.13}$$

5. The mixed variational principle. Determination of the coefficents of Eqs. (3.9),(3.10) describing the perturbed motion of a fluid-containing shell requires knowledge of the vector function $\boldsymbol{\Psi}$, of the proper vibration modes $\mathbf{u}_n$, Φ_n, and of the frequencies ω_n of the fastened fluid-containing shell.

The functions $\boldsymbol{\Psi}$ and Φ_n for arbitrary shells containing fluid volumes which do not admit of complete separation of variables in solving the Helmholtz equations can be determined only approximately. Solution of the problem by variational methods in displacements entails difficulties having to do with satisfying kinematic boundary condition (1.4) at the wet surface of the shell [7].

A Castigliano-type variational principle [8, 9] is an effective means of determining the natural frequencies and modes and also the function $\boldsymbol{\Psi}$ in the case of undeformed cavities containing an incompressible fluid. This principle yields continuity equation (1.5) and kinematic boundary condition (1.4). It is also convenient for computing the vibrations of momentless inertialess liquid-containing shells [10].

Vibration modes requiring allowance for the moment and inertial characteristics of the shell can be computed by a mixed variational principle in which the shell displacements are regarded as independent functions together with the pressure in the fluid. The mixed variational principle was applied to the case of an elastic body with cavities containing an incompressible fluid in [11].

Let us consider the mixed variational principle for computing the natural vibrations of an elastic shell containing a heavy compressible fluid. It is convenient to proceed on the basis of the Lagrange principle with undetermined multipliers in formulating various versions of the mixed variational principle. The undetermined Lagrange multiplier in the equation of continuity of the fluid and in the kinematic boundary condition at the wet surface of the shell is the perturbed pressure in the fluid, which is equal to $\rho\omega^2\Phi$ in the case of potential motion of the fluid during harmonic vibrations.

If the fluid is compressible, the equation of continuity follows directly from Lagrange principle (1.1). All that is necessary in this case is that the work performed by the reactions in retaining kinematic constraints (1.4) at the wet surface of the shell and at the shell edges $f_i(\mathbf{u}) = 0$; (where ($f_i(\ldots)$ is a linear algebraic or differential operator) be added to Eq. (1.1); this work is given by

$$A=\rho\omega^2\iint\limits_{S_0}\Phi\left(\mathbf{u}\mathbf{v}-\frac{\partial\Phi}{\partial\mathbf{v}}\right)dS+\sum_i\int\lambda_i f_i(\mathbf{u})\,dl \tag{5.1}$$

Here λ_i are the reactions of the fastened edges of the shell. They can be represented in terms of linear differential expressions in $\mathbf{u}$ on the basis of static relations; alternatively, they can be regarded as independent unknown functions, which is even more convenient in some cases.

Allowing for potential energy (1.2) and work (5.1), the equation of the variational Lagrange principle with undetermined multipliers in the case of free harmonic vibrations can be written in the form

$$\delta\left\{\Pi_0-\frac{\omega^2}{2}\iint\limits_{S_0}m\mathbf{u}^2dS-\frac{\rho\omega^2}{2}\iint\limits_{S_0}\Phi\mathbf{u}\mathbf{v}dS+\right.$$
$$+\frac{\rho}{2}\iint\limits_{\sigma}\left(g\frac{\partial\Phi}{\partial\mathbf{v}}-\omega^2\Phi\right)\frac{\partial\Phi}{\partial\mathbf{v}}d\sigma+\frac{\rho}{2}\iiint\limits_{\tau}(c^2\Delta\Phi+\omega^2\Phi)\,\Delta\Phi d\tau-$$
$$\left.-\frac{\rho\omega^2}{2}\iint\limits_{S_0}\left(\mathbf{u}\mathbf{v}-\frac{\partial\Phi}{\partial\mathbf{v}}\right)\Phi dS-\sum_i\int\lambda_i f_i(\mathbf{u})\,dl\right\}=0 \tag{5.2}$$

Variational equation (5.2) is not valid if the fluid is incompressible ($c\to\infty$) and if the harmonic character of the function Φ is not a prerequisite. Thus, Eq. (5.2) is valid largely in the range of acoustic vibrations.

Another version of the mixed variational principle which yields the continuity equation for both a compressible and an incompressible fluid can be obtained by expressing the potential energy of compression of the fluid in terms of the pressure, i.e.

$$\Pi=\Pi_0+\frac{\rho g}{2}\iint\limits_{\sigma}\left(\frac{\partial\Phi}{\partial\mathbf{v}}\right)^2d\sigma+\frac{\rho\omega^4}{2c^2}\iiint\limits_{\tau}\Phi^2d\tau \tag{5.3}$$

and by adding the work performed by the pressure in retaining the kinematic constraint (the continuity equation) to expression (5.1); this work is given by

$$A=\rho\omega^2\iint\limits_{S_0}\Phi\left(\mathbf{u}\mathbf{v}-\frac{\partial\Phi}{\partial\mathbf{v}}\right)dS+\rho\omega^2\iiint\limits_{\tau}\Phi\left(\Delta\Phi+\frac{\omega^2}{c^2}\Phi\right)d\tau+\sum_i\int\lambda_i f_i(\mathbf{u})\,dl \tag{5.4}$$

Making use of (5.3) and (5.4), we can now rewrite (1.1) as

$$\delta\left\{\Pi_0-\frac{\omega^2}{2}\iint\limits_{S}m\mathbf{u}^2dS-\frac{\rho\omega^2}{2}\iint\limits_{S_0}\Phi\mathbf{u}\mathbf{v}dS+\right.$$
$$+\frac{\rho}{2}\iint\limits_{\sigma}\left(g\frac{\partial\Phi}{\partial\mathbf{v}}-\omega^2\Phi\right)\frac{\partial\Phi}{\partial\mathbf{v}}d\sigma-\frac{\rho\omega^2}{2}\iiint\limits_{\tau}\left(\Delta\Phi+\frac{\omega^2}{c^2}\Phi\right)\Phi d\tau-$$
$$\left.-\frac{\rho\omega^2}{2}\iint\limits_{S_0}\left(\mathbf{u}\mathbf{v}-\frac{\partial\Phi}{\partial\mathbf{v}}\right)\Phi dS-\sum_i\int\lambda_i f_i(\mathbf{u})\,dl\right\}=0 \tag{5.5}$$

Equations (5.2) and (5.3) make it possible to obtain the fluid displacement potential Φ and the shell displacements $\mathbf{u}$ in the form of independent expansions in given coordinate functions with unknown coefficients. This in turn makes it possible to reduce the problem of determining the natural vibration frequencies and modes of a fluid-containing shell to a system of linear algebraic equations by the Ritz method.

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ON THE CONTACT PROBLEM FOR A HALF-PLANE WITH FINITE ELASTIC REINFORCEMENT

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The state of stress of an elastic rod of finite length not acted on by bending moments and fastened to a semi-infinite plate is considered. The problem has already been investigated [1-3], but only for the simpler case where the load is applied to the rod ends. The present paper concerns the case where the force is applied to the center of the rod. The case where a heat source or a thermoelastic deformation center [4] is present at some point of the elastic half-plane is also considered.

As in the aforementioned studies, the problem is stated in the form of a Prandtl integro-differential equation; methods for solving the latter are the subject of an extensive